Physics 216/116 **Fluid Mechanics for Physicists** **Winter 2017**

Problem Set IV: due Thursday, February 16, 2017

Useful references:
Frank Shu; "Gas Dynamics"
P. Drazin and W Reid; "Hydrodynamic Stability"
And, of course, Landau and Lifshitz

1.) *Tapas*

Each of these questions does not require more than a few lines of calculation. *Don't* make them longer than they need be. Do state your reasoning clearly. Try to do these closed book.

a.) What is the width of the laminar boundary layer at the bottom of a viscous fluid rotating at Ω ?

b.) What is the width of a laminar boundary layer of a stagnation flow?

c.) What shaped eddy is most effective at Rayleigh-Benard convection when $\Omega^2 >> g\alpha\beta$?

d.) How will the strength of a vortex tube evolve when the fluid density within rises linearly in time?

2.) Calculate the flow in a boundary layer in a converging channel between two non-parallel plates. See Landau sections 23, 39. Explain your similarity solution carefully.

3.) Derive the dispersion relation for buoyancy waves in a stably stratified fluid with $\partial S/\partial Z > 0$ and $g = -g\tilde{z}$. These are called internal waves. Take the equilibrium hydrostatic. Show that internal waves are 'backward', i.e. the phase and group velocity can be in opposite directions.

b.) Generalize your analysis of internal waves to include rotation effects, where $\Omega = \Omega\hat{z}$. When are corrections to the dispersion relation due to rotation of significance?

4.) Now consider a rotating fluid which is also compressible and *self-gravitating*. For the latter, include a body force $\underline{f} = -\nabla\phi$ where:

$$\nabla^2\phi = 4\pi G\rho \ .$$

Take $\underline{\Omega} = \Omega\hat{z}$, as usual.

a.) For $\underline{k} = k\hat{z}$, show

$$\omega^2 = k^2 c_s^2 - 4\pi G \rho_0 .$$

Welcome to the Jeans instability! What might be the significance of the marginally stable length scale?

b.) For $\underline{k} = k\hat{x}$, show:

$$\omega^2 = k^2 c_s^2 - 4\pi G \rho_0 + 4\Omega^2 .$$

When are all modes stable?

c.) Why might this result of interest in the context of galaxy structure?

d.) How might Jeans instabilities evolve nonlinearly? Why might cooling effects be important here?

5.) Consider a sheared flow $\underline{v}(z)\hat{x}$ in a stably stratified fluid with $\underline{g} = -g\hat{z}$.

a.) Derive the 2D wave eigenmode equation, called the Taylor-Goldstein equation. This extends the Rayleigh equation from the last problem of Set III.

b.) As before, take $\omega = \omega_r + i\gamma$, and substitute $H = \phi/(V-c)^{1/2}$ where c is the along-stream phase velocity. Multiply the re-scaled Taylor-Goldstein equation by H^* and derive a quadratic form.

c.) Show for $\gamma > 0$, the *Richardson number* must satisfy:

$JN^2/V'^2 < 1/4$, for shear-driven instability.

Here $JN^2/V'^2 = \frac{-g}{\rho}\frac{d\rho}{dz}/V'^2$, is the Richardson number.

d.) What is the physics of the Richardson number criterion? What does it mean, in simple terms?

e.) Extra Credit: Can you give a grunt-and-hand gesture derivation of the Richardson number criterion, based on simple physical ideas of competition of energetics?